Analogical Reasoning Errors in Mathematics at Junior College Level

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In this study we explored the errors due to analogical reasoning made by students from a sample of 69 Year 12 students in a Junior College in Singapore. The data collected included students' responses to a test and post-test interview with selected students. The errors seem to stem from their earlier work in mathematics at lower levels. They wrongly transferred structural information from situations that they had met earlier at secondary level.

Mathematics is essentially deductive. The strength of deductive reasoning lies in the fact that if the premises are true and there are no flaws in the logic, then the conclusion has to be true. On the other hand, analogical reasoning does not always lead to valid conclusions. However, analogical reasoning is very pervasive both in mathematics and in everyday life. Polya (1957) commented that "inference by analogy appears to be the most common kind of conclusion, and it is possibly the most essential kind" (p. 43). Analogical reasoning entails understanding something new by comparing with something that is known (English, 1998); and is generally defined as the transfer of structural information from one system (called the source), to another system (called the target) through mapping relational correspondences between the two systems (English & Sharry, 1996).

Two mathematical problems are analogous to each other if they share a common mathematical structure, and Polya (1957) added that inference by analogy is one of the most essential and widely used problem solving strategies. Analogical reasoning in problem solving involves mapping the relational structure of a known source problem onto a new target problem and using this known structure to help solve the target problem (English, 1998). In utilizing analogical reasoning, it is certainly helpful if we have a large pool of acquired mathematical knowledge: facts and definitions, algorithmic procedures, routine procedures, formerly solved problems and formerly proven theorems. The exercise of analogical reasoning is not merely the memorizing of the solution of previously solved problems. It is definitely not rote learning. It involves directed, purposeful mathematical thinking in determining the structural similarities and relational properties between the source and target problems. English also claimed that in analogical reasoning, students need, first to recognize the common relational structures between two analogous problems, second, to know when and how to make use of it.

One of the errors novice problem solvers make is to focus on the superficial features rather than on the underlying relational structural properties between the source and the target problem (English, 1998). Good problem solvers tend to recall accurately the mathematical structure of source problems, while poor problem solvers may not even notice the analogous structure in the target problem (Silver, 1981). It is a common error among students that they ignore the conditions involved in the source or target problems and hence misuse the analogies. These students do not know when and how to make use of the related structural properties, and how this can be of benefit in solving problems. It is essential that in employing analogical reasoning students have the relevant competencies and knowledge about the rules of discourse in the domain.

While we may be critical of the misapplication of analogical reasoning, Lithner (2003) clarified that reasoning by established experience and the identification of similarities are not by definition low quality reasoning. Being able to pick similarities from experience and work

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examples and classify them is essential to the learning of mathematics. Besides being used in problem solving, analogical reasoning also aids in the development of algebraic abstraction. English and Sharry (1996) contended that students compare similar algebraic examples to extract common relational properties which provide the basis of theory construction. Deeper relational insights will result from further similarity comparisons which are filtered through the theory. This process enables the construction and development of mental models which expresses the observed generalized properties of the sum experiences. The construction of these models provides students with the foundation for future learning. They enable students to uncover additional relational properties as they explore new contexts. English and Sharry also added that in their efforts to identify the important features of a new situation, students will initially rely on salient similarities which are readily retrievable; however the kinds of similarities which the students attend to will change with development into that of shared relational structures.

In this study, we report some errors due to analogical reasoning made by second year students in a Singaporean Junior College (Year 12). Specifically, we examine the analogical errors of second year Junior College students in mathematics. The specific research questions include: What are some of the analogical reasoning errors made by students? What may be the related causes of these errors? We expect that answers to these questions will provide teachers with insights into the reasoning errors behind the mistakes in a student's problem solving strategy. Such knowledge will help teachers anticipate students' errors, bearing in mind the misconceptions at the root, so that they can either pre-empt these errors in instruction, or include questions that might surface such errors in assessment for learning.

Methodology

This is a qualitative study is part of a larger study that was carried out in a Junior College (JC) in Singapore (Year 12 students). A qualitative approach was used, as qualitative studies seek to gain understanding from the observed data and build towards a theory from the intuitive understandings gained (see Merriam, 1998). The participants in this study were 69 second year (Year 12) JC students spread over four different classes. These students had finished the JC curriculum and were preparing for their Preliminary Examinations, which is a school summative assessment at the end of the JC course and in which students are tested on what they are taught over the two years.

There were 28 female and 41 male students in the sample. These students were about 18 years of age and had all taken Additional Mathematics, a harder O-level course (end of Year 10) in their respective secondary schools. There were 39 students from two classes doing both regular advanced level mathematics and Further Mathematics, and students from the other two classes consisted of 30 students doing the regular advanced level Mathematics. Students who study Further Mathematics are usually stronger in mathematics. As such the sample can be considered to be of mixed abilities.

All of the students who participated had to sit for a test which lasted two hours. The written test which was constructed included 15 free response items, covering a range of topics which included proof by mathematical induction, inequalities, transformation of graphs, functions, sequences, calculus, trigonometry, complex numbers, permutations and combinations. These topics were chosen to represent a spread of the A-Level Pure Mathematics syllabus. Questions were constructed to test mathematical reasoning skills, not their ability to manipulate expressions. The test items were carefully selected and checked by an expert in the field. The instrument was piloted with a small group of 6 students (other than those in the sample) and as a result some of the questions were revised. The test was marked and the errors documented. The common errors were then further analyzed in the light of the literature to understand the mathematical reasoning behind these errors. Semi-structured interviews were conducted with 11 selected students based on the solutions they had provided to either confirm, refine or to reject the initial understandings. This required an insight into the students' thought processes and an understanding of the meaning that the students had constructed for themselves. Six of the interviewed students were from the top third, two from the middle and three from the bottom third of the students in the sample.

Results and Discussion

We hereby present some of the students' responses to test items together with the interview extracts when their solutions were discussed at a later stage (all names used are pseudonyms are used). One of the most common occurrences of analogical reasoning errors is the misuse of the distributive property, which is quite common among students at lower levels as well. These errors are still seen in the work of Junior College students. In Question 15(b) of this study (Figure 1), students were given the arguments of complex numbers z_1 and z_2 and were asked to find the argument of the complex number $(z_1 + z_2)$. More than a fifth, (20.3%) of the students had assumed that arg $(z_1 + z_2)$ equalled arg (z_1) + arg (z_2) as in Figure 1. These students had wrongly applied the distributive property to arguments. It was the most common mistake committed by students in that question.

<u>Question 15</u>

Given that z_1 and z_2 are complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1) = \theta$, $\arg(z_2) = \alpha$ where $0 < \theta < \alpha < \pi$.

(a) Find arg $(z_1 + z_2)$ in terms of θ and α.

(b) Show that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1 z_2|\cos(\alpha - \theta)$.

a) $\arg(z_1+z_2) = \arg(z_1) + \arg(z_2) = \theta + \alpha$

Figure 1. Solution to question 15 (a) of test instrument.

Figure 2 illustrates a solution from one student where the error can be attributed to invalid analogical reasoning. In the third line, the student had erroneously applied the distributive property onto an exponential function, and made the error of equating $(x^2-2)^{\frac{1}{2}x}$ to $(x^2)^{\frac{1}{2}x} - 2^{\frac{1}{2}x}$

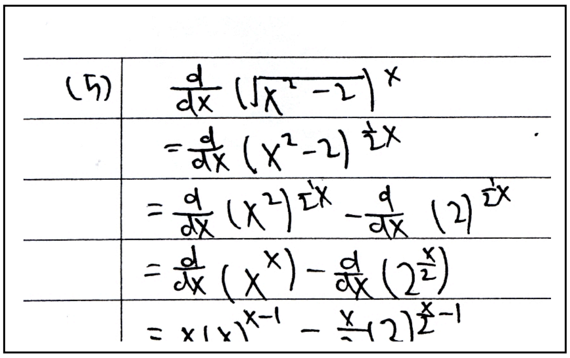

Figure 2. Errors in analogical reasoning.

The last line of Figure 2 also shows an analogical reasoning error, although of a different kind. The student who produced this solution had applied the procedure of differentiating a polynomial to an exponential function. In mimicking the procedure, the student had transferred the solution strategy by identifying similarities (Lithner, 2003). However it was unfortunate that the student had simply focused on the superficial likeness and had not taken into account the structural differences between the two forms of mathematical expressions. In using analogical reasoning, students need to purposefully direct their mathematical thinking to the structural similarities and relational properties between the source and target systems.

In the above examples, we see analogical errors arising from misapplying procedures or rules from one system to another. Analogical reasoning errors can also arise from (or cause) inappropriate symbolic reasoning. Students' response to Question 8(a) is an example (see Figure 3).

<u>Question 8</u>

Integrate (a) $\int_0^{\pi/4} \tan^{-1} x \; dx$

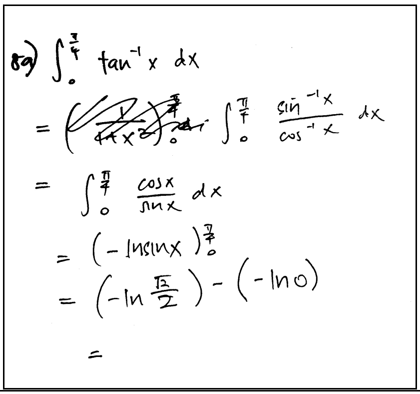

Figure 3. Solution question 8(a) of test instrument.

Figure 3 shows a solution from one of the participants in the pilot study who had thought that $\tan^{-1} x$ equals $\frac{\sin^{-1} x}{\cos^{-1} x}$. The student overlooked the symbol "-1" in this context and thought that it referred to reciprocals. Another student, Ali, asked for the clarification whether $\tan^{-1} x$ equals $\dfrac{\sin^{-1} x}{\cos^{-1} x}$.

I: You have to do by parts.

Ali: Do by sin and cos is it?

I: How would you do by sin and cos?

Ali: Can I clarify if $\tan^{-1} x$ equals $\dfrac{\sin^{-1} x}{\cos^{-1} x}$?

In the following item, the students had to find the volume of the solid generated by a region enclosed between two curves. In the study, 18.8% of the participants made the error of using $\int_0^1 \pi(x - x^2)^2\, dx$. This accounted for almost half of the errors made in the sample. These students who used $\pi\int_a^b [f(x) - g(x)]^2\, dx$ disregarded the difference in shape of the solid formed when the region which is rotated is not bounded by the axes. They did not consider if the hollow centre of the resultant solid would affect the formula used. An example of the error is Figure 4 which is a solution produced by Xiaoli.

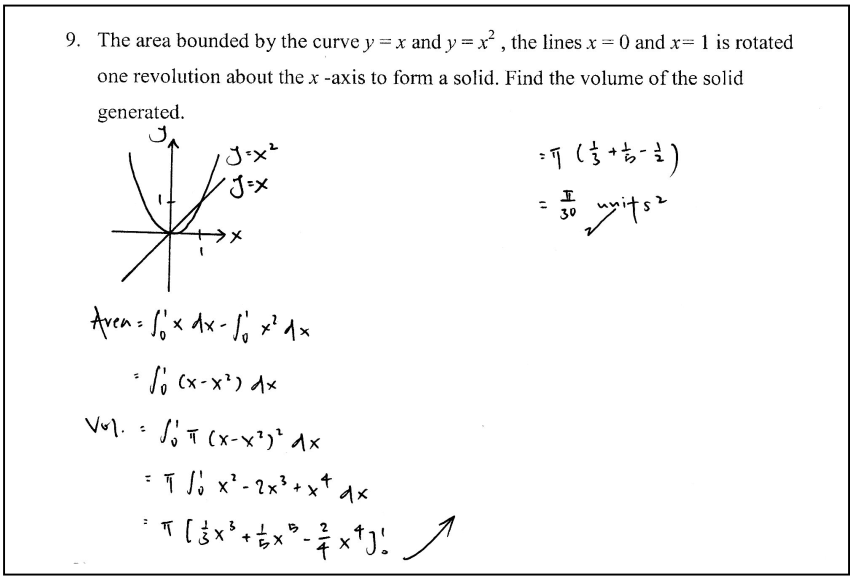

Figure 4. X's solution to Question 9.

Xiaoli had used the formula $\int_0^1 \pi(x - x^2)^2\, dx$ to find the volume generated by rotating the region bounded by the two curves $y = x$ and $y = x^2$ about the x–axis. Subsequently Xiaoli was interviewed:

I: You found the volume by taking $\int_0^1 \pi(x - x^2)^2\, dx$. Why did you do this?

Xiaoli: Because I always feel that volume is generated by area.

I: Do you think the application of the formula like this is correct?

Xiaoli: Ya….Because it says that the area is rotated.

What may be some causes of such errors? Singapore students use integration to find the area of a bounded region in Additional Mathematics in secondary school (Year 10) and proceed on to learn how to calculate the volume of a solid generated by rotating a bounded area about the x or y axis in Junior College. The Additional Mathematics textbooks that the students use in secondary school specifically provide $\int_a^b [f(x) - g(x)]\ dx$ as the formula for finding the area bounded by two curves. However, many students do not realize that the formula came about as a result of the property of integration that $\int_a^b f(x)\ dx - \int_a^b g(x)\ dx = \int_a^b [f(x) - g(x)]\ dx$. They had simply memorised $\int_a^b [f(x) - g(x)]\ dx$ as a formula to use. This could have arisen because students do not know the meaning behind integration. As such, students come to regard $\int_a^b y\ dx$ as the formula for finding area under a curve, rather than see the area under curve as the definition for integration. Students revisit the topic in Junior College, and this time they learn that the integral is defined as the limiting value of the sum area of infinitely many rectangles below and above the curve. They also appreciate the link between differentiation and integration through the Fundamental Theorem of Calculus. However, despite such re-training, students still regard $\int_a^b y\ dx$ as a formula to the area. This illustrates that there is a strong influence from what students learn from classes. When students encounter something which is different, they benchmark it to their learning, and are often unwilling to change. To their schema which already contain $\int_a^b y\ dx$ as a formula to the area, students add the formula $\pi\int_a^b y^2\ dx$ for finding the volume. Even though the proof of the later formula was taught, students perceive it to be unimportant to their preparation to examinations and pay little attention to it. Formulas are detached of their meanings and memorised.

It seems that Xiaoli had looked only at the surface or the perceptual features, which may have in part contributed to the analogical reasoning errors. Students such as Xiaoli who used $\pi\int_a^b [f(x) - g(x)]^2\ dx$ disregarded the difference in shape of the solid formed when the region which is rotated is not bounded by the axes. The fact that the hollow centre of the resultant solid would affect the formula used was not considered.

Analogical reasoning driven solely by perceptions without thought to the underlying structure can lead many into errors. To effectively use analogical reasoning, students not only need to know how to look for common problem structures, they also need to know when and how to make use of them, and how this reasoning can be of benefit in solving the target problem. According to English (1998), having relational knowledge alone is insufficient, conditional knowledge must also be applied. This seems to be a weakness among students. Many do not know the conditions under which a procedure can or cannot be applied.

Some rules such as multiplying by conjugate or rationalising the denominator are also used out of context, as shown in the following problem on complex numbers. Students knew that they need to multiply the numerator and the denominator by the conjugate of the denominator. However, 27.5% of the participants multiplied w by either $\frac{z+\mathbf{i}}{z+\mathbf{i}}$, $\frac{z^*-\mathbf{i}}{z^*-\mathbf{i}}$ or $\frac{z^{-1}-\mathbf{i}}{z^{-1}-\mathbf{i}}$ (see Figure 5).

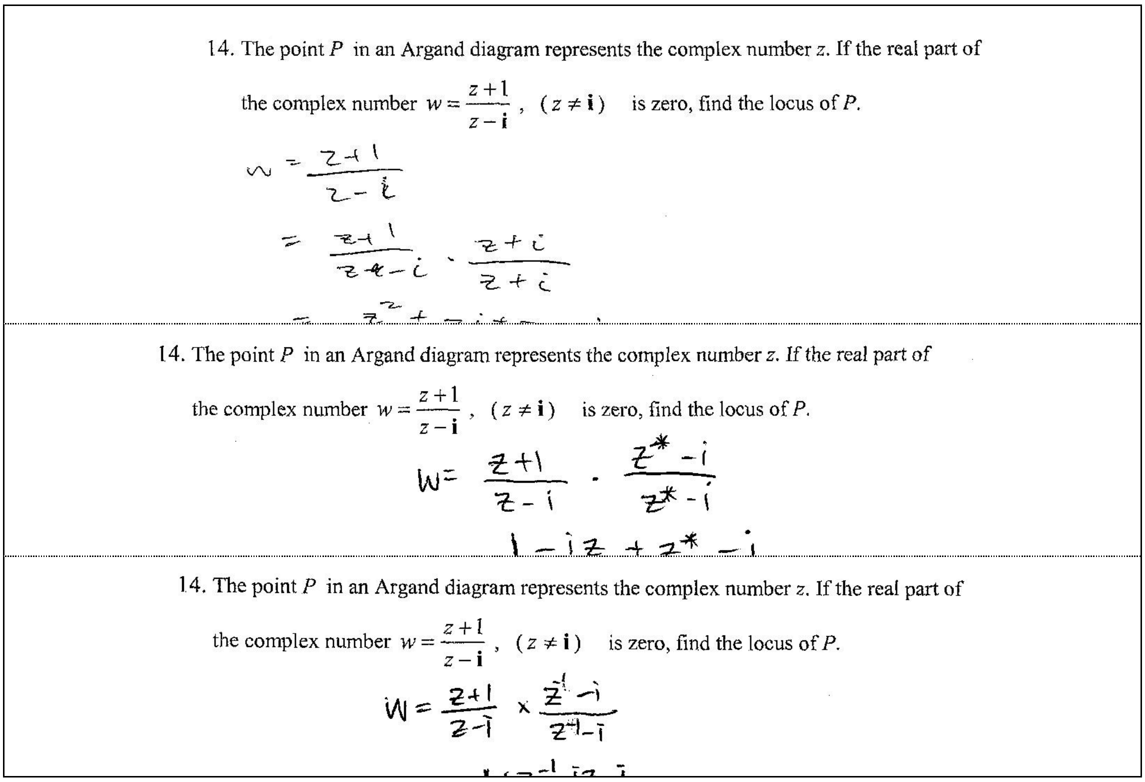

Figure 5. Erroneous solutions to Question 14.

The following is an interview with JT, who produced one of the solutions:

I: What were you trying to do?

JT: I tried to rationalise the denominator.

I: How do you rationalise the denominator?

JT: By trying to make the denominator $a^2 - b^2$, to get rid of the **i**.

The participants did not take into account the **i** in the denominator, or that the z in this question is a complex number and so neither $z + \mathbf{i}$ nor $z^* + \mathbf{i}$ is the conjugate of $z - \mathbf{i}$. Students often miss the conditions in the source or target systems which render familiar, established procedures inapplicable. As we have seen, often analogical reasoning entails using strategies based on established experience (Lithner, 2000), but students need both relational and conditional knowledge to know how and when they can transfer the procedures from the identified situation. An awareness of the conditions and the boundaries in the problem structure will help decide if the procedure one has analogically imported from the source problem is valid or not. Also, the idea of rationalising the denominator when dealing with surds is something learned in Additional mathematics in Year 10. There seems to be an analogy in the way the students use that idea with complex numbers; what JT calls “trying to make the denominator a^2 - b^2”.

Conclusion

The above examples amply demonstrate how instrumental understanding (see Skemp, 1971) at earlier stages influence subsequent learning when concepts are revisited at a later stage. The analogical errors seem to come from the perception of some surface features from the parent situation which are then mapped without any due analysis into a target situation. Besides the lack of relational understanding, common analogical reasoning errors occur because of cognitive discontinuities in the acquisition of mathematical knowledge. Advancement in mathematics learning is never a smooth transition (see Tall, et. al., 2000). In Sfard's (1991) model of the formation of mathematical concepts, processes are reified into mathematical objects which are manipulated in their own right. But sometimes the objects, which have served the students so well in earlier grade levels, fail them when they are applied in new contexts. This is especially so when students do not have a firm grasp of the underlying structure of the mathematical object. In these cases, mathematical ideas, which may have been meaningful in familiar contexts, may no longer be so in new contexts. Moreover, some of the practices that formed the main focus of a student's attention in early exploratory stage must give way and be replaced by notions of generality (English & Sharry, 1996). A student's prior knowledge may sometimes interfere with later learning. Students may have blindly transferred unsuitable features from the source knowledge into a new system.

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